PRACTICE MIDTERM 2 (CHRIST) - SOLUTIONS

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(1a) $\boxed{y' = x^{\ln(x)}\left(\frac{2\ln(x)}{x}\right)}$ (this is just logarithmic differentiation, here are the steps:)

(i) $y = x^{\ln(x)}$

(ii) $\ln(y) = \ln(x)\ln(x) = (\ln(x))^2$

(iii) $\frac{y'}{y} = 2\frac{\ln(x)}{x}$

(iv) $y' = y\left(2\frac{\ln(x)}{x}\right) = x^{\ln(x)}\left(\frac{2\ln(x)}{x}\right)$

(1b) $\boxed{f'(x) = \frac{1}{2\sqrt{arcsin(x)}}\frac{1}{\sqrt{1-x^2}}}$ (this is just the chain rule)

(1c) $\boxed{f'(x) = -2x^{-3}e^{3x} + 3x^{-2}e^{3x}}$ (this is just the chain rule)

(1d) Differentiating, we get: $4x^3 - 3y - 3xy' + 6y'y^2 = 0$, now plugging in $x = 2$ and $y = 1$ and solving for y', we get: $32 - 3 - 6y' + 6y' = 0$, so $29 = 0$, but this is never true, so $\frac{dy}{dx}$ is $\boxed{\text{undefined}}$ at $(2, 1)$

(2a) By using L'Hopital's rule once:

$$\lim_{x\to\infty} \frac{x^{\frac{1}{3}}}{\ln(x)} = \lim_{x\to\infty} \frac{\frac{1}{3}x^{-\frac{2}{3}}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{3}x^{\frac{1}{3}} = \infty$$

(2b) By using L'Hopital's rule twice:

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$$

(3a) (i) Endpoints: $f(-1) = -16$, $f(5) = 20$

(ii) Critical points (1 and 3): $f(1) = 4$, $f(3) = 0$

(iii) Absolute maximum: $\boxed{f(5) = 20}$

(3b) From $xy^2 = 3$, we get $x = \frac{3}{y^2}$, so $9x + y = \frac{27}{y^2} + y$.

(i) Let $f(y) = \frac{27}{y^2} + y$

(ii) The constraint is $y > 0$

(iii) Then $f'(y) = \frac{-54}{y^3} + 1$. And $f'(y) = 0 \Leftrightarrow \frac{-54}{y^3} + 1 = 0 \Leftrightarrow y = \sqrt[3]{54} = 3\sqrt[3]{2}$

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(iv) Also, it's easy to check that $f'(y) > 0$ when $y < 3\sqrt[3]{2}$ and $f'(y) < 0$ when $y < 3\sqrt[3]{2}$. So by the first derivative test for absolute extreme values (section 4.7), it follows that $\boxed{f(3\sqrt[3]{2}) = \frac{3}{2^{\frac{2}{3}}} + 3\sqrt[3]{2}}$ is the absolute minimum of f.

(4a) $y = ax + b$ is a slant asymptote to the graph of f at $-\infty$ if:

$$\lim_{x \to -\infty} f(x) - (ax + b) = 0$$

(4b) $f(c)$ is a local minimum; Cannot conclude anything (this is **not** the same as saying there is no local minimum/maximum)

(4c) $\boxed{L(x) = e^3 + e^3(x - 3)}$

(5) Let $f(x) = \ln(x) - (x-1)$, and suppose that $x > 1$. Then, since f is differentiable on $[1, x]$, by the Mean Value Theorem:

$$\frac{f(x) - f(1)}{x - 1} = f'(c)$$

for some c in $(1, x)$.
Now $f(1) = 0$ and $f'(c) = \frac{1}{c} - 1 < 0$ (since $c > 1$), whence we get:

$$\frac{\ln(x) - (x - 1)}{x - 1} < 0$$

Now multiplying by $x - 1 > 0$, we get:

$$\ln(x) - (x - 1) < 0$$

That is, $\ln(x) < x - 1$